Angular momentum in Einstein-Maxwell theory

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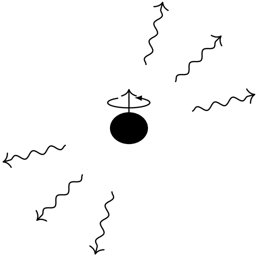
Angular momentum in E&M

- Density of angular momentum stored in field:

$$\vec{\ell} \equiv \vec{r} \times \vec{S}$$

- Why called "angular momentum"? There is a *balance law* (for ℓ_z, say):

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_V \ell_z \mathrm{d}V}_{\text{EM ang. mom.}} + \overset{\text{torque on matter}}{\overset{\downarrow}{\frac{\mathrm{d}L_z}{\mathrm{d}t}}} = - \underbrace{\int_{\partial V} \vec{\mathcal{N}}_z \cdot \mathrm{d}\vec{A}}_{\text{ang. mom. flux}}$$



- Radiated angular momentum $\Longleftrightarrow$ torque on matter

A curiosity

- Flux of energy:

$$\frac{\mathrm{d}E}{\mathrm{d}t} \equiv -\int \vec{S}\cdot\hat{r}\,\mathrm{d}A \sim \int r^2(E_\theta B_\phi - B_\theta E_\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

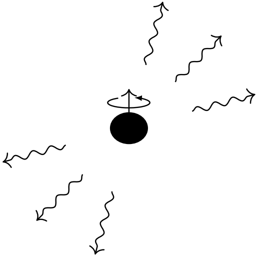
  $E_\theta, E_\phi, B_\theta, B_\phi$ are *radiative*: $\sim\ 1/r$

- Flux of angular momentum:

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} \equiv -\int \vec{\mathcal{N}}_z\cdot\hat{r}\,\mathrm{d}A \sim \int r^3(E_r E_\phi + \cancel{B_r} B_\phi)\sin^2\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

  E_r is *non-radiative/Coulombic*: $\sim\ 1/r^2$!

- What is the significance of this difference?



[Ashtekar & Bonga, 2017], problem 9.8 of [Jackson, 1999]

Outline

I. Conserved Currents in Electromagnetism

II. Electromagnetism near null infinity

III. Asymptotic charges in Einstein-Maxwell theory

Outline

Stress-energy currents

- $\ell_z, \vec{\mathcal{N}}_z$ components of a conserved *four-current* $j_{\mathrm{T}}^a(\partial_\phi)$:

$$\frac{\partial \ell_z}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{N}}_z = 0 \iff \nabla_a j_{\mathrm{T}}^a(\partial_\phi) = 0$$

- Special case of *stress-energy current* constructed from ξ^a and stress-energy tensor T_{EM}^{ab}:

$$j_{\mathrm{T}}^a(\boldsymbol{\xi}) \equiv T_{\mathrm{EM}}^{ab} \xi_b,$$

  [$j_{\mathrm{T}}^a(\partial_t)$ gives energy]

- Flux-balance law:

$$\nabla_a (T_{\mathrm{EM}}^{ab} + T_{\mathrm{matter}}^{ab}) = 0$$

  Tied *directly* to behavior of matter fields!

Uniqueness of this current

- Defining feature of this current: conserved for any symmetry ξ^a, in absence of matter:

$$\nabla_a j^a_{\mathrm{T}}(\boldsymbol{\xi}) = T^{ab}_{\mathrm{EM}} \underbrace{\cancel{\nabla_{(a}\xi_{b)}}}_{\xi^a \text{ a symmetry}} + \xi_a \overbrace{\cancel{\nabla_b T^{ab}_{\mathrm{EM}}}}^{\text{no matter}}$$

- "Conserved current associated with symmetry" not unique prescription!

$$j^a_{\mathrm{T}}(\boldsymbol{\xi}) \to j^a_{\mathrm{T}}(\boldsymbol{\xi}) + \nabla_b Q^{ab}(\boldsymbol{\xi})$$

  still conserved, for any antisymmetric $Q^{ab}(\boldsymbol{\xi})$

- Variety of currents of this form, useful for different purposes

Example: Noether current

- Varying Lagrangian gave T^{ab}_{EM}, but can give other currents:

$$\delta(\sqrt{-g}L_{\mathrm{EM}}) = \sqrt{-g}\left[\frac{1}{2}\boxed{T^{ab}_{\mathrm{EM}}}\,\delta g_{ab} + \frac{1}{4\pi}\left(\nabla_b F^{ba}\right)\delta A_a + \nabla_a \boxed{\theta^a_{\mathrm{EM}}(\delta\boldsymbol{A})}\right]$$

- *Symplectic potential* $\theta^a_{\mathrm{EM}}(\delta\boldsymbol{A})$:

$$\theta^a_{\mathrm{EM}}(\delta\boldsymbol{A}) = \frac{1}{4\pi}F^{ab}\delta A_b$$

- Noether current $j^a_{\mathrm{N}}(\boldsymbol{\xi})$ conserved:

$$j^a_{\mathrm{N}}(\boldsymbol{\xi}) \equiv \theta^a_{\mathrm{EM}}(\pounds_\xi \boldsymbol{A}) - L_{\mathrm{EM}}\xi^a$$

- Note: $\pounds_\xi A^a$ variation of A^a under symmetry

Example: Canonical current

- Symplectic current:

$$\sqrt{-g}j^a_{\mathrm{S}}(\delta_1\boldsymbol{A},\delta_2\boldsymbol{A})\equiv\delta_1[\sqrt{-g}\theta^a_{\mathrm{EM}}(\delta_2\boldsymbol{A})]-1\longleftrightarrow 2$$

- j^a_{S} conserved for *linearized solutions* $\delta_1 A_a, \delta_2 A_a$
  $\implies$ works for *arbitrary* solutions (EoM *linear*)
- Note: A_a solution & ξ^a symmetry $\implies$ $\pounds_\xi A_a$ *also* solution
- Canonical current:

$$j^a_{\mathrm{C}}(\boldsymbol{\xi})\equiv j^a_{\mathrm{S}}(\boldsymbol{A},\pounds_\xi\boldsymbol{A})$$

Relation to stress-energy current

- Relation for j^a_{N} known for general theories [Iyer & Wald, 1994]:

$$j^a_{\mathrm{N}}(\boldsymbol{\xi}) = -j^a_{\mathrm{T}}(\boldsymbol{\xi}) - \nabla_b \underbrace{Q^{ab}_{\mathrm{N}}(\boldsymbol{\xi})}_{\text{“Noether charge”}}$$

- Canonical current trickier:

$$j^a_{\mathrm{C}}(\boldsymbol{\xi}) = 2 j^a_{\mathrm{N}}(\boldsymbol{\xi}) - \nabla_b Q^{ab}_{\mathrm{C}}(\boldsymbol{\xi}) + \underbrace{K^a_{\mathrm{C}}(\boldsymbol{\xi})}_{\text{vanishes for exact symmetries}}$$

Uses for currents (not exhaustive!)

- Stress-energy: systems involving generic matter (unknown Lagrangian)—e.g., usual classical electromagnetism!
- Canonical (and symplectic generally):
  - First law of black hole mechanics [e.g., Iyer & Wald, 1994]
  - Positive energy for stability [e.g., Hollands & Wald, 2012]
  - Conserved currents from "hidden symmetries" [e.g., G & Flanagan, 2019 & 2020]
  - Self force? [G & Moxon, "in prep."]
- Noether: asymptotic charges (later in this talk)

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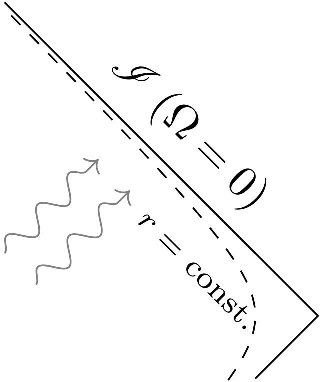
Null infinity

- Studying behavior of fields far from a source
- *Conformally compactify* spacetime to bring infinity to a finite place:

$$\underbrace{\mathrm{d}s^2}_{\text{“unphysical”}} = \Omega^2 \underbrace{\mathrm{d}\hat{s}^2}_{\text{“physical”}}, \qquad \Omega \sim 1/r$$

- “Null infinity”: endpoint of null geodesics becomes submanifold $\mathscr{I}$, where $\Omega = 0$
- $\mathscr{I}$ a *null manifold*:
  - Normal $n_a \equiv \nabla_a \Omega$ is null
  - Intrinsic degenerate metric q_{ab} ($n^a q_{ab} = 0$)
  - (Non-unique) inverse q^{ab} defined given null l_a s.t. $l_a n^a = -1$



For a review, see [Geroch, 1977]

Bondi-Metzner-Sachs (BMS) group: Asymptotic symmetries

- Symmetry ψ of null infinity: preserves structure of $\mathscr{I}$, up to rescaling:

$$\psi : (q_{ab}, n^a) \mapsto (\omega^2 q_{ab}, \omega^{-1} n^a)$$

- Conserved currents associated with infinitesimal symmetry generators ξ^a
- ξ^a splits (given l^a):

$$\xi^a = \underbrace{q^a{}_b X^b}_{\text{“Lorentz part”}} + \overbrace{\beta n^a}^{\text{“supertranslation part”}}$$

- Note: exact symmetries limit to these ξ^a on $\mathscr{I}$!
  - Rotations $\to X^a$
  - Translations $\to \beta n^a$ with $\beta = \sum_{l=0,1} \beta_{lm} Y_{lm}$

Electromagnetic fields near null infinity

F_{ab} and A_a split into "radiative" and "non-radiative" pieces

- Radiative:

$$\mathcal{A}_a \equiv q_a{}^b A_b, \qquad \underbrace{\mathcal{E}_a \equiv q_a{}^b F_{bc} n^c}_{E_\theta, E_\phi, B_\theta, B_\phi} = -\pounds_n \mathcal{A}_a$$

- (Relevant) non-radiative:

$$\underbrace{\mathrm{Re}[\varphi_1] \equiv \frac{1}{2} l^a n^b F_{ab}}_{E_r}$$

Restating the unusual property of angular momentum

Flux of current $j^a(\boldsymbol{\xi})$ through portion of $\mathscr{I}$:

$$\mathcal{F}[\Delta\mathscr{I}, \boldsymbol{\xi}] = \int_{\Delta\mathscr{I}} (\cdots)\, \mathrm{d}\mathscr{I}$$

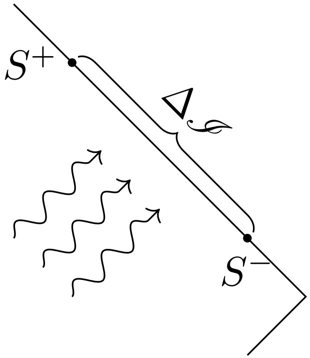
Examples for $j^a_{\mathrm{T}}(\boldsymbol{\xi})$:

- Supermomentum:

$$\mathcal{F}_{\mathrm{T}}[\Delta\mathscr{I}, \beta\boldsymbol{n}] \sim \int_{\Delta\mathscr{I}} \beta\mathcal{E}_a\mathcal{E}^a\, \mathrm{d}\mathscr{I}$$

- Angular momentum:

$$\mathcal{F}_{\mathrm{T}}[\Delta\mathscr{I}, \boldsymbol{X}] \sim \int_{\Delta\mathscr{I}} \mathrm{Re}[\varphi_1]\mathcal{E}_a X^a \mathrm{d}\mathscr{I}$$



Other currents

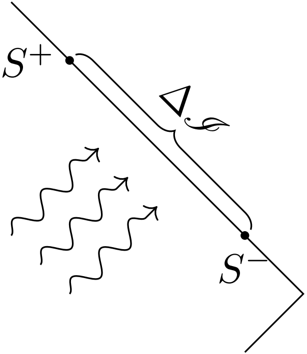
- Noether current:

$$\mathcal{F}_{\mathrm{N}}[\Delta\mathscr{I}, \boldsymbol{\xi}] \sim \int_{\Delta\mathscr{I}} \mathcal{E}^a \pounds_\xi \mathcal{A}_a \, \mathrm{d}\mathscr{I}$$

- Canonical current:

$$\mathcal{F}_{\mathrm{C}}[\Delta\mathscr{I}, \boldsymbol{\xi}] \sim \int_{\Delta\mathscr{I}} q^{ab} (\mathcal{E}_a \pounds_\xi \mathcal{A}_a - \mathcal{A}_a \pounds_\xi \mathcal{E}_b - \mathcal{E}_a \mathcal{A}_b \pounds_n \beta) \, \mathrm{d}\mathscr{I}$$

Both only contain radiative degrees of freedom!



Boundary terms

- Freedom in conserved currents changes fluxes:

$$j^a(\boldsymbol{\xi}) \to j^a(\boldsymbol{\xi}) + \nabla_b Q^{ab}$$

$$\Downarrow$$

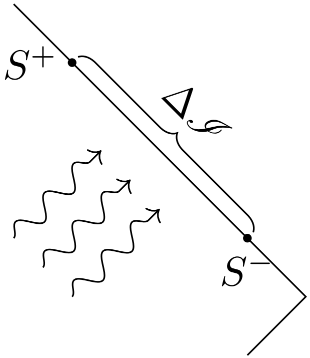
$$\mathcal{F}[\Delta\mathscr{I}, \boldsymbol{\xi}] \to \mathcal{F}[\Delta\mathscr{I}, \boldsymbol{\xi}] + \mathcal{Q}[S, \boldsymbol{\xi}]\Big|_{S_-}^{S_+}$$

- Fluxes related by

$$\mathcal{F}_{\mathrm{N}}[\Delta\mathscr{I}, \boldsymbol{\xi}] = -\mathcal{F}_{\mathrm{T}}[\Delta\mathscr{I}, \boldsymbol{\xi}] - \mathcal{Q}_{\mathrm{N}}[S, \boldsymbol{\xi}]\Big|_{S_-}^{S_+}$$

$$= \frac{1}{2}\left\{\mathcal{F}_{\mathrm{C}}[\Delta\mathscr{I}, \boldsymbol{\xi}] + \mathcal{Q}_{\mathrm{C}}[S, \boldsymbol{\xi}]\Big|_{S_-}^{S_+}\right\}$$

$$\mathcal{Q}_{\mathrm{N}}[S, \boldsymbol{\xi}] \sim \int_S \mathrm{Re}[\varphi_1]\mathcal{A}_a X^a \, \mathrm{d}S, \qquad \mathcal{Q}_{\mathrm{C}}[S, \boldsymbol{\xi}] \sim \int_S \beta \mathcal{E}^a \mathcal{A}_a \, \mathrm{d}S$$



- Note: $\mathcal{Q}_{\mathrm{N}}[S, \boldsymbol{X}] \neq 0$, even for $S \to S_{\pm\infty}$

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What is a charge?

Example: electromagnetic charge

- Two ways to calculate this charge:

$$Q = \int_V \rho \mathrm{d}V \sim \int_{\partial V} \vec{E} \cdot \mathrm{d}\vec{A}$$

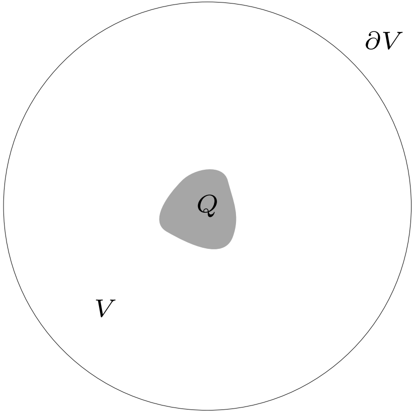
- Asymptotic fields give charge:

$$E_r \sim \frac{Q}{r^2} + \dots$$

- Consequence of

$$J^a = \frac{1}{4\pi} \nabla_b F^{ab}$$

- What about other conserved quantities?

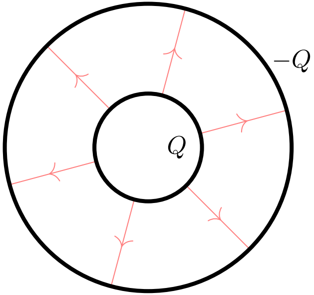


Energy/angular momentum as a charge?

Same with L_z? Energy E?

$$E \sim \int_V \vec{E}^2 \, \mathrm{d}V \stackrel{?}{=} \int_{\partial V} (\cdots) \cdot \mathrm{d}\vec{A}$$

No!



$$\vec{E} \neq 0 \implies \int_V \vec{E}^2 \, \mathrm{d}V \neq 0$$

$$\vec{E} \text{ vanishes outside } \implies \int_{\partial V} (\cdots) \cdot \mathrm{d}\vec{A} = 0$$

Charges in GR

Stationary, axisymmetric metric:

$$\mathrm{d}s^2 = -\left[1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right]\mathrm{d}t^2 - \left[\frac{4L_z}{r}\sin^2\theta + O\left(\frac{1}{r^2}\right)\right]\mathrm{d}t\mathrm{d}\phi$$
$$+ \left[1 + O\left(\frac{1}{r}\right)\right]\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]$$

$$\Longrightarrow \qquad g_{tt} = -1 + 2M/r + \ldots, \qquad g_{t\phi} = -\frac{4L_z}{r}\sin^2\phi + \ldots$$

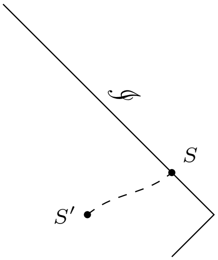
- Asymptotic fields give M & L_z—*GR charges*
- Note: non-stationary metrics similar, but M & L_z *change* in time

Alternative approach: Komar formula

- Consider exact symmetry ξ^a: $\nabla_a \xi_b = \nabla_{[a}\xi_{b]}$ antisymmetric
- Define "Faraday tensor" $F^{\xi}_{ab} \equiv \nabla_a \xi_b$; "Komar formula" given by

$$Q_\xi \sim \lim_{S' \to S} \int_{S'} \vec{E}^{\xi} \cdot \mathrm{d}\vec{S}\,'$$

- Issues with this approach:
  - $Q_{\partial_t} = M$, but $Q_{\partial_\phi} = -2L_z$
  - Generalizing exact $\to$ asymptotic symmetries complicates definition, yielding "linkages" (e.g., [Geroch and Winicour, 1981])

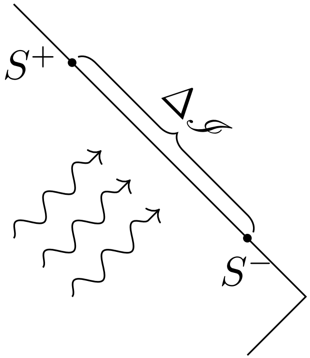


Wald-Zoupas (WZ) procedure

- Given Lagrangian for some theory, gives charges $\mathcal{Q}[S, \boldsymbol{\xi}]$ for any asymptotic symmetry ξ^a
- Fully covariant—doesn't rely on expansions of the metric in particular coordinate systems
- Fluxes *only* depend on radiative degrees of freedom:

$$\mathcal{F}[\Delta\mathscr{I}, \boldsymbol{\xi}] \equiv \mathcal{Q}[S, \boldsymbol{\xi}]\Big|_{S^-}^{S^+}$$

- "Noether's theorem for radiation": Symmetries $\implies$ "conserved quantities"
- In vacuum GR, $\mathcal{Q}_{\rm GR}[S, \boldsymbol{\xi}]$ matches Komar formula with correct factors



[Wald & Zoupas, 1999]

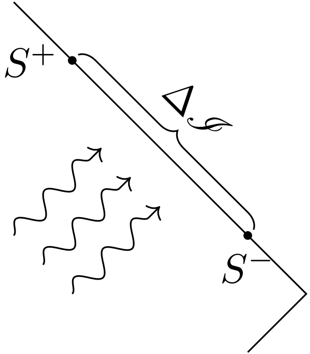
Curious property of angular momentum in Einstein-Maxwell

- Consider $\mathcal{Q}_{\mathrm{GR}}[S, \boldsymbol{\xi}]$
  - In vacuum:

$$\mathcal{Q}_{\mathrm{GR}}[S, \boldsymbol{\xi}]\Big|_{S^-}^{S^+} = \underbrace{\mathcal{F}_{\mathrm{GR}}[\Delta\mathscr{I}, \boldsymbol{\xi}]}_{\text{only radiative d.o.f.}}$$

  - With E&M:

$$\mathcal{Q}_{\mathrm{GR}}[S, \boldsymbol{\xi}]\Big|_{S^-}^{S^+} = \mathcal{F}_{\mathrm{GR}}[\Delta\mathscr{I}, \boldsymbol{\xi}] - \underbrace{\mathcal{F}_{\mathrm{T}}[\Delta\mathscr{I}, \boldsymbol{\xi}]}_{\text{contains non-radiative d.o.f.}}$$



- WZ charge *must* only on radiative d.o.f.
  $\implies$ $\mathcal{Q}_{\mathrm{GR}}[S, \boldsymbol{\xi}]$ *not* WZ charge for Einstein-Maxwell!

Two approaches for asymptotic charges w/ matter

1. WZ with only L_{GR}
   - \+ charges can be determined entirely from GR d.o.f.
   - – fluxes contain non-radiative d.o.f. (potentially)

2. WZ with $L = L_{\mathrm{GR}} + L_{\mathrm{matter}}$
   - \+ in some sense the “true” WZ charges, fluxes only contain radiative d.o.f.
   - – requires knowledge of L_{matter}, charges depend on matter d.o.f. as well

Approaches yield same answer for simple scalar fields
$\Longrightarrow$ Einstein-Maxwell provides simplest case where they *differ*

WZ in Einstein-Maxwell

- WZ procedure for charge is difficult, but flux nicely factors:

$$\mathcal{F}[\Delta\mathscr{I},\boldsymbol{\xi}] = \mathcal{F}_{\mathrm{GR}}[\Delta\mathscr{I},\boldsymbol{\xi}] + \mathcal{F}_{\mathrm{EM}}[\Delta\mathscr{I},\boldsymbol{\xi}],$$

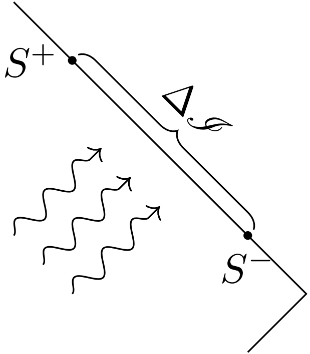
where

$$\mathcal{F}_{\mathrm{EM}}[\Delta\mathscr{I},\boldsymbol{\xi}] \sim \underbrace{\int_{\Delta\mathscr{I}} \mathcal{E}^a \pounds_{\xi} \mathcal{A}_a \,\mathrm{d}\mathscr{I}}_{\mathcal{F}_{\mathrm{N}}[\Delta\mathscr{I},\boldsymbol{\xi}]!}$$

- Can solve for $\mathcal{Q}[S,\boldsymbol{\xi}] = \mathcal{Q}_{\mathrm{GR}}[S,\boldsymbol{\xi}] + \mathcal{Q}_{\mathrm{EM}}[S,\boldsymbol{\xi}]$:

$$\mathcal{Q}[S,\boldsymbol{\xi}]\Big|_{S^-}^{S^+} = \mathcal{F}_{\mathrm{GR}}[\Delta\mathscr{I},\boldsymbol{\xi}] + \mathcal{F}_{\mathrm{N}}[\Delta\mathscr{I},\boldsymbol{\xi}]$$

$$= \underbrace{\mathcal{F}_{\mathrm{GR}}[\Delta\mathscr{I},\boldsymbol{\xi}] - \mathcal{F}_{\mathrm{T}}[\Delta\mathscr{I},\boldsymbol{\xi}]}_{\mathcal{Q}_{\mathrm{GR}}[S,\boldsymbol{\xi}]\big|_{S^-}^{S^+}} \underbrace{-\mathcal{Q}_N[S,\boldsymbol{\xi}]}_{\therefore \mathcal{Q}_{\mathrm{EM}}[S,\boldsymbol{\xi}]}\Big|_{S^-}^{S^+}$$

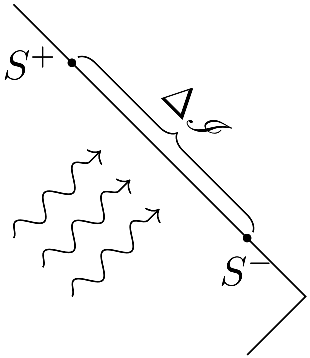


Properties of these charges

- E&M piece of the charge depends on E&M d.o.f.:

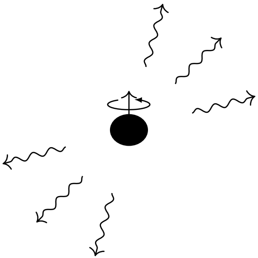
$$\mathcal{Q}_{\text{EM}}[S,\boldsymbol{\xi}] \sim \int_S \text{Re}[\varphi_1]\mathcal{A}_a X^a$$

- $\mathcal{Q}_{\text{EM}}[S,\beta\boldsymbol{n}] = 0$ (vanishes for supertranslations) $\implies M$ unchanged, but L_z differs from vacuum GR



Examples

- Kerr-Newman spacetime: black hole with spin parameter a, mass M, charge Q
  - L_z different than usual Ma?
  - Nope!
- Spinning charged sphere [varying $\omega(t)$]
  - Net torque on system: $\mathcal{F}_{\mathrm{T}}[\Delta\mathscr{I}, \partial_\phi] \neq 0$ [Bonga, Poisson, Yang; 2018]
  - L_z has E&M corrections



Conclusions & Future Work

- Angular momentum in E&M:
  - Flux of usual (stress-energy) definition contains non-radiative data
  - Alternative definitions of angular momentum are purely radiative
  - No "correct" conserved quantities associated with symmetries
  - Different notions useful for different problems
- Asymptotic charges in theories w/ matter
  - Use vacuum GR charges, or do full Wald-Zoupas?
  - Difference in Einstein-Maxwell for angular momentum from non-radiative data
  - Full Wald-Zoupas requires matter Lagrangian—not always possible!
- Future work:
  - Three currents considered here: how are they related for other theories?
  - Other theories with more dramatic difference b/w charge procedures?